

Multidimensional β -skeletons in L_1 and L_∞ metric ^{*}

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Abstract. The β -skeleton $\{G_\beta(V)\}$ for a point set V is a family of geometric graphs, defined by the notion of neighborhoods parameterized by real number $0 < \beta < \infty$. By using the distance-based version definition of β -skeletons we study those graphs for a set of points in \mathbb{R}^d space with l_1 and l_∞ metrics. We present algorithms for the entire spectrum of β values and we discuss properties of lens-based and circle-based β -skeletons in those metrics. Let $V \in \mathbb{R}^d$ in L_∞ metric be a set of n points in general position. Then, for $\beta < 2$ lens-based β -skeleton $G_\beta(V)$ can be computed in $O(n^2 \log^d n)$ time. For $\beta \geq 2$ there exists an $O(n \log^{d-1} n)$ time algorithm that constructs β -skeleton for the set V . We show that in \mathbb{R}^d with L_∞ metric, for $\beta < 2$ β -skeleton $G_\beta(V)$ for n points can be computed in $O(n^2 \log^d n)$ time. For $\beta \geq 2$ there exists an $O(n \log^{d-1} n)$ time algorithm. In \mathbb{R}^d with L_1 metric for a set of n points in arbitrary position β -skeleton $G_\beta(V)$ can be computed in $O(n^2 \log^{d+2} n)$ time.

1 Introduction

β -skeletons [15] in R^2 belong to the family of proximity graphs, geometric graphs in which two vertices (points) define an edge if and only if they satisfy particular geometric requirements.

β -skeletons are both important and popular because of many practical applications which span a spectrum of areas including geographic information systems, wireless ad hoc networks and machine learning. They allow to reconstruct a shape of a two-dimensional object from a given set of sample points and they are also helpful in finding the minimum weight triangulation of a point set. Two different forms of β -neighborhoods have been studied for $\beta > 1$ (see [2,6]) leading to two different families of β -skeletons: lens-based and circle-based ones.

Well-known examples of lens-based β -skeletons include *Gabriel Graph* (1-beta skeleton), defined by Gabriel and Sokal [9] and *Relative Neighborhood Graph RNG* (for $\beta = 2$) which was introduced by Toussaint [23].

Many algorithms computing Gabriel graphs and relative neighborhood graphs in sub-quadratic time were created [19,22,10,12,18,13]. Some of them can be also used in metrics different than euclidean (L_p metrics for $1 < p < \infty$). In L_1 and L_∞ metrics the problem of computing 2-dimensional *RNG* has been addressed by Lee [17] and O'Rourke [20]. Wulff-Nilsen [24] presented an algorithm that computes Gabriel graph in the plane with a fixed orientation metric. Maignan and Gruau [8] defined metric Gabriel graphs which can be

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used in arbitrary metric space.

The concept of constructing β -skeletons in higher-dimensional space is still not well explored. Toussaint [23] presented an algorithm that computes the relative neighborhood graph for a set of n points in \mathbb{R}^d in $O(n^3)$ time. Supowit [22] designed the first subcubic algorithm that computes the *RNG* in $O(n^2)$ but only in case where no three points in the input set form an isosceles triangle. His method is based on partitioning the space around each point from the given set so that a linear size supergraph of *RNG* can be constructed. From a straightforward edge elimination of this supergraph a relative neighborhood graph can be computed. It has been shown that this algorithm extends to L_p metrics (see [7,25]). Based on those ideas Smith [21] provided an algorithm that computes the *RNG* in \mathbb{R}^3 for a set of n points in general position in $O(n^{\frac{23}{12}} \log n)$ time. Jaromczyk and Kowaluk [10] improved those results to obtain an $O(n^2)$ algorithm that works in L_p , $1 < p < \infty$ metrics for a set of points in general position. They also provided an algorithm [11] that computes the relative neighborhood graph for an arbitrary set of n points in \mathbb{R}^3 in $O(n^2 \log n)$.

An algorithm presented by Katajainen and Nevalainen [16] can be used in d -dimensional spaces in L_p metrics for $1 \leq p \leq \infty$. In 3-dimensional euclidean space it works in $O(n^{\frac{5}{2+\epsilon}})$ time.

Agarwal and Matoušek [1] used data structures based on the partitioning scheme of Chazelle et al. [4] and an arrangement of spheres (see [5]) to prove that for a set of n points in general position in \mathbb{R}^3 the relative neighborhood graph can be constructed in $O(n^{\frac{3}{2} + \epsilon})$, for every $\epsilon > 0$. They also provided an algorithm that computes *RNG* of an arbitrary set of points in \mathbb{R}^3 in $O(n^{\frac{7}{4+\epsilon}})$.

O'Rourke [20] designed an algorithm computing the relative neighborhood graph for a set of n points in L_∞ in general position in $O(n^2 \log n)$ time. Smith [21] shed that in any \mathbb{R}^d space with L_∞ the *RNG* can be constructed in $O(n(\log n)^{d-1})$ time, assuming that the points are in general position.

This paper is organized as follows. The basic properties and definitions for β -skeletons in \mathbb{R}^d space with L_1 and L_∞ metric are presented in section 2. In section 3 we describe algorithms computing β -skeletons in L_∞ . In the next section we give an algorithm for constructing those graphs in L_1 metric. The last section contains open problems and conclusions.

2 Preliminaries

Let V be a set of n points in \mathbb{R}^d space with L_1 (or respectively L_∞) metric. For any two points $p_1, p_2 \in \mathbb{R}^d$ we define the set $S(p_1, p_2)$ which is the sum of all shortest paths between p_1 and p_2 , where point p belongs to $S(p_1, p_2)$ if and only if $d_1(p_1, p) + d_1(p, p_2) = d_1(p_1, p_2)$ ($d_\infty(p_1, p) + d_\infty(p, p_2) = d_\infty(p_1, p_2)$ respectively).

By modifying the definition of lens-based β skeletons [15] we receive a definition based only on a distance criterion:

Definition 1. For a given set of points V in \mathbb{R}^d space with a L_1 (or L_∞) metric and for parameters $0 \leq \beta \leq \infty$ we define a graph $G_\beta(V)$ - called a lens-based β -skeleton - as follows: two points v_1 and v_2 are connected with an edge if and only if at least one lens in $N_1(v_1, v_2, \beta)$ (or $N_\infty(v_1, v_2, \beta)$ respectively) does not contain points from $V \setminus \{v_1, v_2\}$ where:

- for $0 < \beta < 1$ a lens $N_1(v_1, v_2, \beta)$ (or $N_\infty(v_1, v_2, \beta)$ respectively) is the intersection of two discs, each of them has radius $\frac{d_1(v_1, v_2)}{2\beta}$ ($\frac{d_\infty(v_1, v_2)}{2\beta}$ respectively), whose boundaries contain both v_1 and v_2 and each shortest path connecting their centers intersects $S(v_1, v_2)$,
- for $1 \leq \beta < \infty$ a lens $N_1(v_1, v_2, \beta)$ (or $N_\infty(v_1, v_2, \beta)$, respectively) is the intersection of two discs with radius $\frac{\beta d_1(v_1, v_2)}{2}$ ($\frac{\beta d_\infty(v_1, v_2)}{2}$, respectively) and with centers c_1 and c_2 , respectively, such that $d_1(v_1, c_2) = d_1(v_2, c_1) = \frac{|\beta - 2|d_1(v_1, v_2)|}{2}$, $d_1(v_1, c_1) = d_1(v_2, c_2) = \frac{|\beta d_1(v_1, v_2)|}{2}$ ($d_\infty(v_1, c_2) = d_\infty(v_2, c_1) = \frac{|\beta - 2|d_\infty(v_1, v_2)|}{2}$, $d_\infty(v_1, c_1) = d_\infty(v_2, c_2) = \frac{|\beta d_\infty(v_1, v_2)|}{2}$ in L_∞ metric) and each shortest path connecting c_1 and c_2 intersects $S(v_1, v_2)$.

Note that the shortest path intersection condition from the definition for $0 < \beta < 1$ and for $1 \leq \beta < \infty$ ensures that centers of the discs defining the lens lie antipodically in relation to the set $S(v_1, v_2)$.

The definition [6] of the circle-based β -skeletons can also be modified in the similar way:

Definition 2. [6] For a given set of points V in \mathbb{R}^d space with a L_1 (or L_∞) metric and for parameters $0 \leq \beta < \infty$ we define a graph $G_\beta(V)$ - called a circle-based β -skeleton - as follows: two points v_1 and v_2 are connected with an edge if and only if at least one lens in $N_1^c(v_1, v_2, \beta)$ (or $N_\infty^c(v_1, v_2, \beta)$, respectively) does not contain points from $V \setminus \{v_1, v_2\}$ where:

1. for $\beta < 1$ we define set $N_1^c(v_1, v_2, \beta)$ (or $N_\infty^c(v_1, v_2, \beta)$ respectively) the same way as for lens-based β -skeleton;
2. for $1 \leq \beta < \infty$ a lens $N_1^c(v_1, v_2, \beta)$ (or $N_\infty^c(v_1, v_2, \beta)$ respectively) is the sum of two discs, each of them has radius $\frac{\beta d_1(v_1, v_2)}{2}$ ($\frac{\beta d_\infty(v_1, v_2)}{2}$, respectively), whose boundaries contain both v_1 and v_2 and each shortest path connecting their centers intersects $S(v_1, v_2)$.

In our further considerations we will use the following lemma:

Lemma 1. Let $C_1^d(p, r)$ ($C_\infty^d(p, r)$, respectively) denote a sphere in \mathbb{R}^d space with L_1 (L_∞ , respectively) metric centered in a point p and with the radius r . In d -dimensional L_∞ space, for two points v_1, v_2 $S(v_1, v_2) = \bigcup_{0 \leq r \leq d_i(v_1, v_2)} C_i^d(v_1, r) \cap C_i^d(v_2, d_i(v_1, v_2) - r)$, where $i = 1, \infty$.

Proof. It is obvious that for each point $p \in \bigcup_{0 \leq r \leq d_i(v_1, v_2)} C_i^d(v_1, r) \cap C_i^d(v_2, d_i(v_1, v_2) - r)$, where $i = 1, \infty$ there is $d_i(v_1, v_2) = d_i(v_1, p) + d_i(p, v_2)$, i.e. $p \in S(v_1, v_2)$. If $p \in S(v_1, v_2)$, there exists $r = d_i(v_1, p)$, where $i = 1, \infty$, such that $d_i(p, v_2) = d_i(v_1, v_2) - r$. Then $p \in C_i^d(v_1, r) \cap C_i^d(v_2, d_i(v_1, v_2) - r)$, i.e. $p \in \bigcup_{0 \leq r \leq d_i(v_1, v_2)} C_i^d(v_1, r) \cap C_i^d(v_2, d_i(v_1, v_2) - r)$.

Let s_k^d be a line in \mathbb{R}^d with L_∞ metric containing k -th diagonal of the cube $[0, 1]^d$, where $k \in \{1, \dots, 2^{d-1}\}$. Then $S(v_1, v_2)$ is a polyhedron whose edges are parallel to some line s_k^d (see Figure 1). Let $T_k(v_1, v_2)$ be a sum of all lines parallel to s_k^d intersecting $S(v_1, v_2)$ and $T(v_1, v_2) = \bigcup_k T_k(v_1, v_2)$. $T(v_1, v_2)$ is a cross with 2^d arms (see Figure 1).

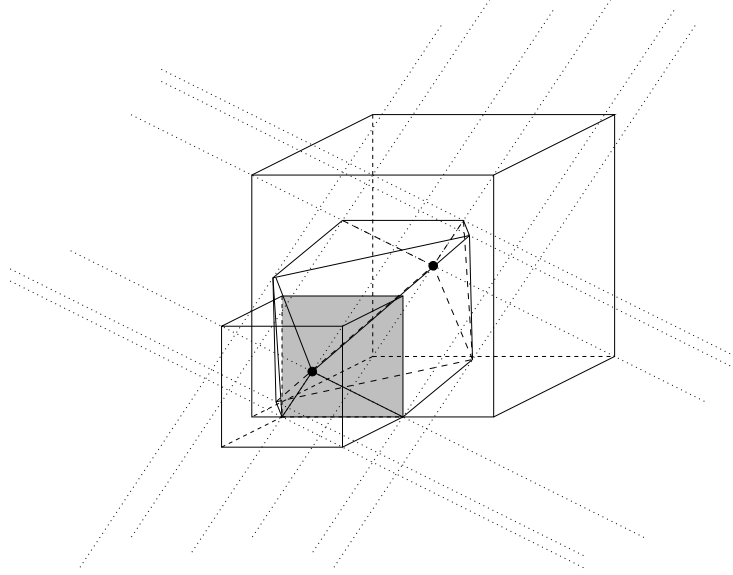


Fig. 1. Intersection of two spheres in L_∞ (grey square), the set $S(v_1, v_2)$ (lines in bold) and two sets $T_{k_1}(v_1, v_2)$ and $T_{k_2}(v_1, v_2)$ (dotted lines).

In L_1 metric the set $S(v_1, v_2)$ is a cuboid (at most d -dimensional). It is a center of the cross $T(v_1, v_2)$ whose each of d arms $T_k(v_1, v_2)$ is parallel to some coordinate axis (see Figure 2).

Lemma 2. For $0 < \beta < 1$ (and for $\beta > 1$ for circle-based β -skeletons) and points v_1, v_2 , let c_1, c_2 be such points that $d_1(c_1, v_1) = d_1(c_2, v_1) = d_1(c_1, v_2) = d_1(c_2, v_2) = \frac{d_1(v_1, v_2)}{2\beta}$ ($d_\infty(c_1, v_1) = d_\infty(c_2, v_1) = d_\infty(c_1, v_2) = d_\infty(c_2, v_2) = \frac{d_\infty(v_1, v_2)}{2\beta}$, respectively). Then c_1 and c_2 define a lens $N_1(v_1, v_2, \beta)$ ($N_\infty(v_1, v_2, \beta)$, respectively) for v_1, v_2 if and only if both c_1 and c_2 belong to the set $T_k(v_1, v_2)$ for some direction $k \in \{1, \dots, 2^{d-1}\}$ ($k \in \{1, \dots, d\}$, respectively).

Proof. Because a distance between c_1 and c_2 is greater than $d_1(v_1, v_2)$ ($d_\infty(v_1, v_2)$, respectively) then $c_1, c_2 \notin S(v_1, v_2)$. From the definition of β -skeleton it follows that $S(c_1, c_2)$ is divided by $S(v_1, v_2)$ into two separate parts. Hence, there exists an edge direction of $S(c_1, c_2)$ (e.g. s_k^d) that $S(v_1, v_2)$ divides $T_k(c_1, c_2)$. Hence $T_k(c_1, c_2) \subset T_k(v_1, v_2)$, i.e. $c_1, c_2 \in T_k(v_1, v_2)$.

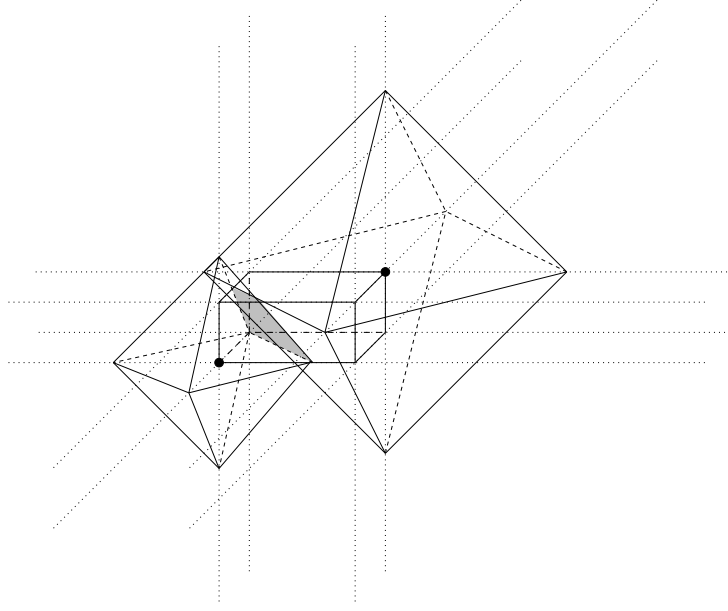


Fig. 2. Intersection of two spheres in L_1 (grey area), the set $S(v_1, v_2)$ (lines in bold) and all sets $T_{k_i}(v_1, v_2)$, where $i \in \{1, 2, 3\}$ (dotted lines).

3 β -skeletons in L_∞ metric

We will concentrate on 3-dimensional space to better describe the solution. Let V be a set of n points in 3-dimensional L_∞ space. Let $v_1 = (x_1, y_1, z_1), v_2 = (x_2, y_2, z_2)$.

For $\beta = 1$ and for points v_1 and v_2 let c_1 and c_2 be the centers of the discs that define a lens for v_1, v_2 . Depending on how many numbers from $|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|$ is equal to the distance $d_\infty(v_1, v_2)$ the intersection of spheres $C(v_1, \frac{d_\infty(v_1, v_2)}{2})$ and $C(v_2, \frac{d_\infty(v_1, v_2)}{2})$ can be a rectangle, a segment or a point.

For each point $v \in C(v_1, \frac{d_\infty(v_1, v_2)}{2}) \cap C(v_2, \frac{d_\infty(v_1, v_2)}{2})$ there is $d_\infty(v, v_1) + d_\infty(v, v_2) = d_\infty(v_1, v_2)$, so this all such points belong to $S(v_1, v_2)$.

Let c_1, c_2 be two points from the intersection set such that the distance between them is maximal. In each case there is at most two pairs of points like that (in rectangle both pairs of such points define the same lens). Then the lens defined by c_1 and c_2 is minimal.

For a given $\beta \in (1, 2)$, let C_1 (C_2 respectively) be the intersection of spheres $C(v_1, \frac{\beta d_\infty(v_1, v_2)}{2})$ and $C(v_2, (1 - \frac{\beta}{2})d_\infty(v_1, v_2))$ ($C(v_2, \frac{\beta d_\infty(v_1, v_2)}{2})$ and $C(v_1, (1 - \frac{\beta}{2})d_\infty(v_1, v_2))$ respectively). Note that just like for $\beta = 1$ set C_1 (C_2 respectively) can be a rectangle, a segment or a point depending on how many numbers from $|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|$ is equal to the distance $d_\infty(v_1, v_2)$. Sets C_1 and C_2 are symmetrical and parallel to each other and also parallel to one or more of the $\{x = 0\}, \{y = 0\}$ or $\{z = 0\}$ planes.

Note that since for any point v in the set $C(v_1, \frac{\beta d_\infty(v_1, v_2)}{2}) \cap C(v_2, (1 - \frac{\beta}{2})d_\infty(v_1, v_2))$ ($C(v_2, \frac{\beta d_\infty(v_1, v_2)}{2}) \cap C(v_1, (1 - \frac{\beta}{2})d_\infty(v_1, v_2))$ respectively) we have $d_\infty(v, v_1) + d_\infty(v, v_2) = \frac{\beta d_\infty(v_1, v_2)}{2} + (1 - \frac{\beta}{2})d_\infty(v_1, v_2) = d_\infty(v_1, v_2)$ and $\frac{\beta d_\infty(v_1, v_2)}{2} \leq d_\infty(v_1, v_2)$. Hence, each point from the set C_1 (C_2 respectively) also belongs to the set $S(v_1, v_2)$. A pair $c_1 \in C_1, c_2 \in C_2$ defines the minimal lens if the distance between them is maximal. In some cases there can be more than one such pair, but all those pairs define the same lens.

If $\beta = 2$, the lens for points v_1, v_2 is defined uniquely.

For $0 < \beta < 1$ in case of the lens-based β -skeleton (or for $\beta > 1$ for circle-based ones), all possible centers of the discs defining the lenses belong to the intersection of two spheres $C(v_1, \frac{d_\infty(v_1, v_2)}{2\beta})$ and $C(v_2, \frac{d_\infty(v_1, v_2)}{2\beta})$. If segment $v_1 v_2$ is not parallel to a coordinate axis, then this intersection is a curve composed of six edges of the cuboid $D(v_1, \frac{d_\infty(v_1, v_2)}{2\beta}) \cap D(v_2, \frac{d_\infty(v_1, v_2)}{2\beta})$, where $D(p, r) = \{q \in \mathbb{R}^3 : d_\infty(p, q) \leq r\}$. In other case, the intersection is not a curve, but a band.

Note that for $\beta = 1$ the intersection of spheres that defines centers of lenses is a rectangle that is equal to a cross-section of the set $S(v_1, v_2)$. For $1 > \beta \rightarrow 0$ (or $1 < \beta \rightarrow \infty$ for circle-based skeletons) vertices of this rectangle travel away from the set $S(v_1, v_2)$ along the lines parallel to diagonals of a 3-dimensional cube. For $\beta = 1$ vertices of the curve $C(v_1, \frac{d_\infty(v_1, v_2)}{2\beta}) \cap C(v_2, \frac{d_\infty(v_1, v_2)}{2\beta})$ are in the set $S(v_1, v_2)$ and that for $\beta \rightarrow 0$ ($1 < \beta \rightarrow \infty$, respectively) they move further away from $S(v_1, v_2)$ inside arms of the set $T(v_1, v_2)$. Therefore, the intersection of the curve $C(v_1, \frac{d_\infty(v_1, v_2)}{2\beta}) \cap C(v_2, \frac{d_\infty(v_1, v_2)}{2\beta})$ and set $T(v_1, v_2)$ are respective vertices of the cuboid $D(v_1, \frac{d_\infty(v_1, v_2)}{2\beta}) \cap D(v_2, \frac{d_\infty(v_1, v_2)}{2\beta})$ along with short segments, parts of the intersection curve, incident to those vertices. If c_1, c_2 are endpoints of a diagonal of the cuboid $D(v_1, \frac{d_\infty(v_1, v_2)}{2\beta}) \cap D(v_2, \frac{d_\infty(v_1, v_2)}{2\beta})$ and they are both in the curve $C(v_1, \frac{d_\infty(v_1, v_2)}{2\beta}) \cap C(v_2, \frac{d_\infty(v_1, v_2)}{2\beta})$ then the lens they define is minimal.

For $\beta > 1$ in the case of circle-based skeleton we compute the pairs of closest centers c_1 and c_2 . It can be done in constant time. We also test emptiness of both spheres centered in c_1 and c_2 instead of only their intersection.

If segment $v_1 v_2$ is parallel to an axis then the intersection of the band that contains the potential centers of the spheres that define lenses and the arms of the set $T(v_1, v_2)$ is a sum of eight segments, four on each side of the band.

For $\beta > 2$ let us consider the set C_1 (C_2 respectively) which is the intersection of spheres $C(v_1, \frac{\beta d_\infty(v_1, v_2)}{2})$ and $C(v_2, (1 - \frac{\beta}{2})d_\infty(v_1, v_2))$ ($C(v_2, \frac{\beta d_\infty(v_1, v_2)}{2})$ and $C(v_1, (\frac{\beta}{2} - 1)d_\infty(v_1, v_2))$ respectively). Note that since $(\frac{\beta}{2} - 1)d_\infty(v_1, v_2) + d_\infty(v_1, v_2) = \frac{\beta}{2}d_\infty(v_1, v_2)$ then spheres $C(v_1, \frac{\beta d_\infty(v_1, v_2)}{2})$ and $C(v_2, (1 - \frac{\beta}{2})d_\infty(v_1, v_2))$ ($C(v_2, \frac{\beta d_\infty(v_1, v_2)}{2})$ and $C(v_1, (\frac{\beta}{2} - 1)d_\infty(v_1, v_2))$ respectively) are tangent internally and the sphere with the smaller radius is contained in the bigger one. From this we get that the set C_1 (C_2 respectively) is a sum of up to three respective faces of the sphere with the smaller radius, depending on the number of

coordinates that define the distance $d_\infty(v_1, v_2)$. Intersection of C_1 (C_2 respectively) and $T(v_1, v_2)$ is a subset of vertices of C_1 (C_2 respectively).

Based on the observations in this section the following theorem can be proved:

Theorem 1. *Let $V \in \mathbb{R}^3$ in L_∞ metric be a set of n points in general position. Then, for $\beta < 2$ β -skeleton $G_\beta(V)$ can be computed in $O(n^2 \log^3 n)$ time. For $\beta \geq 2$ there exists an $O(n \log^2 n)$ time algorithm.*

Proof. Since all lenses in L_∞ metric are cuboids, range search algorithms [21,3] can be used to compute β -skeletons. Note that for $\beta < 2$ it is necessary to check all $O(n^2)$ edges, but for each edge the minimal lens is uniquely defined. For $\beta \geq 2$, since $G_\beta(V)$ is a subgraph of the euclidean relative neighborhood graph, then $G_\beta(V)$ has $O(n)$ edges. For each edge there is a constant $O(C)$ number of minimal lenses, so the lens-based β -skeleton can be computed in $O(n \log^2 n)$ time [21].

Used the same methods we obtain the following result.

Theorem 2. *Let $V \in \mathbb{R}^d$ in L_∞ metric be a set of n points in general position. Then, for $\beta < 2$ β -skeleton $G_\beta(V)$ can be computed in $O(n^2 \log^d n)$ time. For $\beta \geq 2$ there exists an $O(n \log^{d-1} n)$ time algorithm.*

For $\beta' > \beta$ there is $G_{\beta'}(V) \subset G_\beta(V)$. Moreover, for a pair of points v_1, v_2 and $i < \infty$ the following inclusion holds $N_i(v_1, v_2, \beta) \subset N_\infty(v_1, v_2, \beta)$. Due result of Agarwal and Matousek [1] we obtain the following theorem.

Theorem 3. *Let $V \in \mathbb{R}^3$ in L_∞ metric be a set of n points in arbitrary position. Then, for $\beta \geq 2$ and each $\epsilon > 0$ there exists an $O(n^{\frac{7}{4+\epsilon}} \log^d n)$ time algorithm that constructs β -skeleton for the set V .*

4 β -skeletons in L_1 metric

Now, let us consider the problem for a set V of n points in \mathbb{R}^d with the L_1 metric. We will concentrate on 3-dimensional space to better describe our considerations. Let $v_1 = (x_1, y_1, z_1), v_2 = (x_2, y_2, z_2)$.

For $\beta < 1$ (and for $\beta > 1$ in case of circle-based β -skeletons) and for points $v_1 = (x_1, y_1, z_1), v_2 = (x_2, y_2, z_2)$ the shape of the intersection $C(v_1, \frac{d_1(v_1, v_2)}{2\beta})$ and $C(v_2, \frac{d_1(v_1, v_2)}{2\beta})$ depends on whether $|x_1 - x_2| = |y_1 - y_2|$. If $|x_1 - x_2| \neq |y_1 - y_2|$ this intersection is a parallelogram which common part with the cross $T(v_1, v_2)$ consists of two pairs of parallel segments. For each pair two most distant points define the minimal lens.

If $|x_1 - x_2| = |y_1 - y_2| \neq 0$ then set $C(v_1, \frac{d_1(v_1, v_2)}{2\beta}) \cap C(v_2, \frac{d_1(v_1, v_2)}{2\beta})$ consists of two parallel segments. Cross $T(v_1, v_2)$ intersects each of those segments and a pair of most distant centers, one from one segment and one from another, defines the minimal lens.

In the last case, when $|x_1 - x_2| = |y_1 - y_2| = 0$ the intersection is a square. Set $T(v_1, v_2)$ intersects it in four points but the lenses for two pairs of centers defined this way are identical.

For $\beta \in [1, 2)$ there are also three cases to consider. If segment v_1v_2 is parallel to some coordinate axis then the intersection of spheres $C(v_1, \frac{\beta d_1(v_1, v_2)}{2})$ and $C(v_2, \frac{(\beta-2)d_1(v_1, v_2)}{2})$ (spheres $C(v_2, \frac{\beta d_1(v_1, v_2)}{2})$ and $C(v_1, \frac{(\beta-2)d_1(v_1, v_2)}{2})$ respectively) is just one point. From the definition, it lies in the set $S(v_1, v_2)$ and the lens for points v_1, v_2 is uniquely defined.

If $|z_1 - z_2| = 0$ but segment v_1v_2 is not parallel to any coordinate axis then the intersection is a single segment in the plane $\{z = z_1\}$. This segment is fully contained in the cuboid $S(v_1, v_2)$. Since the intersection of spheres $C(v_2, \frac{\beta d_1(v_1, v_2)}{2})$ and $C(v_1, \frac{(\beta-2)d_1(v_1, v_2)}{2})$ is a segment parallel to this one, that also lies in the cuboid $S(v_1, v_2)$, the pair of points c_1 and c_2 , one from one segment, one from the other, define the minimal lens if the euclidean distance between them is maximal. Since the segments described above are symmetrical and parallel it is only necessary to consider their endpoints.

If $|z_1 - z_2| \neq 0$ and v_1v_2 is not parallel to a coordinate axis then the intersection $C(v_1, \frac{\beta d_1(v_1, v_2)}{2}) \cap C(v_2, \frac{(\beta-2)d_1(v_1, v_2)}{2})$ is a hexagon with edges pairwise parallel. Note that since for any point v in this hexagon it is true that $d_1(v, v_1) + d_1(v, v_2) = \frac{\beta d_1(v_1, v_2)}{2} + \frac{(\beta-2)d_1(v_1, v_2)}{2} = d_1(v_1, v_2)$, so this polygon is contained in the set $S(v_1, v_2)$. The intersection of spheres $C(v_2, \frac{\beta d_1(v_1, v_2)}{2})$ and $C(v_1, \frac{(\beta-2)d_1(v_1, v_2)}{2})$ is also a symmetrical hexagon, parallel to the previous one, that is also in $S(v_1, v_2)$. A pair of points c_1, c_2 one from one hexagon, second from the other such that the euclidean distance between them is maximal define the minimal lens. There can be up to six pairs like that.

For $\beta = 2$ the centers of the discs defining the lenses are uniquely defined.

For $\beta > 2$ let us denote by C_1 (C_2 respectively) the intersection of spheres $C(v_1, \frac{\beta d_1(v_1, v_2)}{2})$ and $C(v_2, \frac{(\beta-2)d_1(v_1, v_2)}{2})$ ($C(v_2, \frac{\beta d_1(v_1, v_2)}{2})$ and $C(v_1, \frac{(\beta-2)d_1(v_1, v_2)}{2})$ respectively). C_1 can be equal to either one face of the sphere with the smaller radius or to a sum of two or even four such faces. If points v_1 and v_2 have all coordinates different ($|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2| \neq 0$) then this intersection is one of the faces from sphere with radius $\frac{\beta d_1(v_1, v_2)}{2}$. Set $T(v_1, v_2)$ intersects this faces in three points (in its vertices). Each such point belongs to a different arm of $T(v_1, v_2)$. In set C_2 there are also only three points, so for each pair of opposing arms exists a pair c_1, c_2 that defines a lens.

If v_1 and v_2 have one of the coordinates equal then C_1 is a sum of two respective faces. The cross intersects it in vertices of those two triangles. Therefore, there are 4 pairs of lenses defined for v_1, v_2 .

In the last case, when segment v_1v_2 is parallel to one of the coordinates axis, the intersection $C(v_1, \frac{\beta d_1(v_1, v_2)}{2}) \cap C(v_2, \frac{(\beta-2)d_1(v_1, v_2)}{2})$ ($C(v_2, \frac{\beta d_1(v_1, v_2)}{2}) \cap C(v_1, \frac{(\beta-2)d_1(v_1, v_2)}{2})$ respectively) is a sum of four neighbouring faces. Only vertices of those faces belong to arms of the set $T(v_1, v_2)$. There are five points in set C_1 (C_2 respectively) and therefore there

are 5 lenses defined for v_1, v_2 . Note that in each case the pair $c_1 \in C_1, c_2 \in C_2$ such that the distance between them is maximal defines the smallest lens.

The algorithm constructing β -skeleton for a given parameter β is similar to the previous one. We use range trees to test emptiness of lenses.

However, in this case analyzed regions are not cuboids. Therefore, we modify range tree structure by adding two new coordinates corresponding to each pair directions of sphere faces in L_1 . In this way we obtain a finite number of data structures. We test intersections of spheres analyzing pairs of their parts each of which is limited by planes parallel to the coordinate axes and one plane parallel to face of the sphere.

Based on the observations in this section the following theorem can be proved:

Theorem 4. *Let $V \in \mathbb{R}^d$ in L_1 metric be a set of n points in arbitrary position. Then, β -skeleton $G_\beta(V)$ can be computed in $O(n^2 \log^{d+2} n)$ time.*

5 Conclusions and open problems

We have described shapes of lenses for β -skeletons in \mathbb{R}^d space with L_1 and L_∞ metric. The definition we used is based on the distance criterion and on the idea of choosing pairs of antipodal points by computing the sets of shortest paths between them. Since for each pair $v_1, v_2 \in V \subseteq \mathbb{R}^d$ and for every β there is no more then a constant number of minimal lenses (and this constant number does not depend on the number of points in the set V) it is possible to use range search algorithm to compute the β -skeleton. Since in euclidean metric in \mathbb{R}^3 a size of Gabriel Graph is $O(n^{\frac{3}{2}})$ [14], we think that the algorithm for $1 \leq \beta \leq 2$ in L_∞ metric presented in this work could be improved. Moreover, algorithms in L_1 seem to be not optimal.

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